RELATIVE MOTION AND INERTIAL RANGE OF PARTICLES

A. T. Litvinov

Inzhenerno-Fizicheskii Zhurnal, Vol. 10, No. 6, pp. 776–782, 1966

UDC 541.182

An examination is made of the relative motion and inertial range of particles in the region Re ≤ 300 with a drag coefficient of the medium that takes inertial terms into account in deceleration of the particles.

In solving a number of theoretical problems it is necessary to determine the rate of acceleration of a particle or liquid drop injected into a gas stream, the length of the acceleration section, the relative velocity of the particle at any point onward from the point of injection into the stream, and also the inertial range.

We shall examine the relative motion of a particle introduced into a gas stream in the region Re ≤ 300, and to this end we shall make a number of conventional assumptions—that the particle is spherical, that the forces of hydrodynamic interaction between particles and the force of gravity are negligibly small, and that the stream flow is uniform and steady.

It is known that the drag coefficient during accelerated motion of a particle is greater than in uniform motion, and, of course, smaller in decelerated motion.

The drag force in nonuniform motion of a spherical particle in the region Re < 1 may be determined from the equation [1]

$$\bar{F}_d = 3\pi\eta d\bar{v}_0 + \frac{1}{12}\pi d^3\rho_2 \frac{d\bar{v}}{dt} + + 0.75 d^3 \sqrt{\pi\eta\rho_1}\int_0^t \frac{d\bar{v}}{dt}\frac{dx}{\sqrt{t-x}}. \quad (1)$$

The first term on the right of (1) takes account of the resistance of the medium at constant relative velocity (instantaneous velocity at a given instant), while the second expresses the drag force due to the particle dragging behind it some volume of the viscous medium (for a spherical particle the apparent additional mass is equal to half the volume of the medium displaced by the particle). The inertial force of the additional mass must be overcome by the particle in its motion. The third term allows for the force expended in setting the additional mass in motion. This integral term is usually neglected, without giving any basis for the simplification.

At increased values of Reynolds number the influence of the inertial terms—the second and third in (1)—increases, especially for nonuniform motion of the particle. Therefore, neglecting them leads to considerable error.

To partially allow for the inertial terms numerous empirical equations have been proposed describing the functional dependence of the drag coefficient ψ on Re number, i.e., $\psi = f(\mathrm{Re}, v_0)$ [3–8].

When a particle (or a liquid drop) is introduced into a stream, it is entrained by the stream because of the hydrodynamic force, and the relative velocity between the particle and the gas stream decreases as the particle accelerates.

Using experimental data [8], we shall express the drag coefficient during the decelerating change of relative velocity ($v_0 \to 0$) by the following expression [9]:

$$\psi = A + B/\mathrm{Re}, \quad (2)$$

where A and B are constants, equal, respectively, to 0.12 and 37 for the region 2 < Re ≤ 300. Values of ψ calculated from (2) agree satisfactorily with the experimental values of [8] in the region 5 < Re ≤ 300 (Fig. 1) obtained in decelerating relative motion, i.e., with allowance for inertia terms.

For more accurate calculations it is necessary to take the following values of the constants: a) for the region 100 < Re ≤ 300 A = 0.055 and B = 50, b) for 6 < Re ≤ 100 0.205 and 37, respectively, and c) for 0 < Re ≤ 6 4.45 and 24.

The change in momentum for a particle injected into a stream is

$$m\frac{d\bar{v}}{dt} = \bar{F}_d. \quad (3)$$

The particle velocity is

$$\bar{v} = \frac{dx}{dt} = \bar{v}_g - \bar{v}_0. \quad (4)$$

The drag force of the medium in the region Re ≤ ≤ 300 is

$$\bar{F}_d = \left(A + \frac{B}{\mathrm{Re}}\right)\frac{\pi d^2}{8}\rho_1(\bar{v}_g - \bar{v})\,|\bar{v}_g - \bar{v}|. \quad (5)$$

Substituting the values of the force from (5) and of the coefficient from (2) into (3), we obtain a differential equation for the particle motion:

$$\frac{\pi d^3}{6}\rho_2\frac{d\bar{v}}{dt} = \left(A + \frac{B}{\mathrm{Re}}\right)\frac{\pi d^2}{8}\rho_1|\bar{v}_0|^2. \quad (6)$$

Differentiating (4) with respect to time, we obtain

$$\frac{d\bar{v}}{dt} = -\frac{d\bar{v}_0}{dt}, \quad (7)$$

since $\bar{v}_g$ = const.

Substituting the value of $d\bar{v}/dt$ from (7) into (6), and making appropriate transformations, we write (6) as follows:

$$\frac{d\bar{v}_0}{dt} = -(\alpha v_0^2 + \beta v_0), \quad (8)$$

where $\alpha = 0.75\,A\rho_1(d\rho_2)^{-1}$, $\beta = 0.75\,B\eta(d^2\rho_2)^{-1}$.

Multiplying both sides of (8) by dx, and making appropriate transformations, we obtain

$$dx = -dv_0(v_r - v_0)/(\alpha v_0^2 + \beta v_0). \quad (9)$$

Integrating (9) in the range $x_1 = 0$ (i.e., the point of injection of the particle into the stream) to $x_2 = l_e$, and the relative velocity from $v_0 = v_g$ to $v_0 = v_x$, we obtain

$$l_e = \int_{v_x}^{v_g} \frac{dv_0(v_g - v_0)}{\alpha v_0^2 + \beta v_0} =$$

$$= \left| \left(\frac{v_g}{\beta} - \frac{1}{\alpha}\right) \ln(\alpha v_0 + \beta) - \frac{v_g}{\beta} \ln v_0 \right|_{v_x}^{v_g} =$$

$$= Dv_g d^2 \rho_2 \left[\left(1 + \frac{C}{v_g}\right) \ln \frac{v_g + C}{v_x + C} - \ln \frac{v_g}{v_0}\right], \quad (10)$$

where $D = 1.33/B\eta$; $C = B\nu(Ad)^{-1}$.

Analysis of Eq. (10) shows that the acceleration path length l_e of the particle is proportional to the gas stream velocity v_g, and to the second power of its diameter d and density ρ_2, i.e., the smaller the particle, the more quickly it is entrained and the shorter its entrainment path length (Fig. 2).

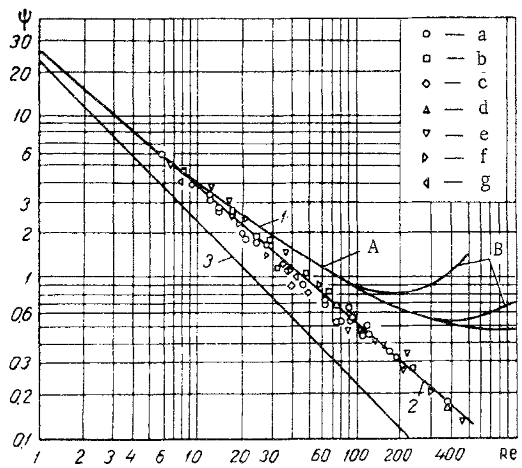

Fig. 1. Dependence of drag coefficient of medium, ψ, on Re number: 1) at constant flow velocity v_0 = constant (A—solid spheres, B—freely falling water drops); 2) experimental data at v = 30, 42, and 54 m/sec and T = 27° C for water, isooctane, and trichloroethylene (*a*, e, and *f*, respectively), at v = 54 m/sec and T = 4° C for water (b), at 54 m/sec and 27° C for MgO (c), at 30 and 42 m/sec and 27° C for CaSi (d), 54 m/sec and 370° C for trichloroethylene (g); 3) with $\psi = 24/\mathrm{Re}$ (Stokes law).

As may be seen from the figure, a particle with $d = 10^{-6}$ m, $\rho_2 = 2000$ kg/m^3 in a medium with viscosity $\eta = 1.83 \cdot 10^{-5}$ N · sec/m^2 is entrained by the gas stream (v_g = 80 m/sec) in a length $l_e \sim 10^{-3}$ m. Particles with $d = 10^{-5}$ and 10^{-4} m under the same conditions are entrained in lengths of $l_e \sim 9.45 \cdot 10^{-2}$ and 6.05 m, respectively. Numerical values of the acceleration distance calculated from the equation given in [5], for particles with $d = 10^{-6}$, 10^{-5}, and 10^{-4} m in similar conditions are approximately $8 \cdot 10^{-4}$, $7 \cdot 10^{-2}$, and 3.08 m, i.e., the higher the value of Re, the greater the error.

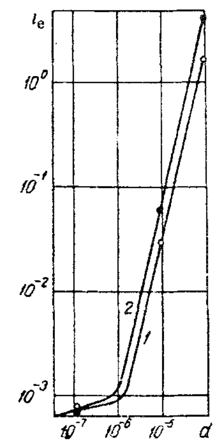

Fig. 2. Dependence of particle acceleration path length l_e (m) on diameter d (m): 1) according to the equation given in [5] (with partial allowance for inertia terms); 2) according to Eq. (10).

Separating the variables and integrating (7) over the same range of variation of relative velocity as in (10), we obtain

$$t = -\int_{v_x}^{v_g} \frac{dv_0}{\alpha v_0^2 + \beta} = \gamma \ln \left| \frac{(v_g + C) v_x}{(v_x + C) v_g} \right|, \quad (11)$$

where $\gamma = 1.33\, d^2 \rho_2 (B\eta)^{-1}$.

The equations obtained, (10) and (11), make it possible, by the method of successive approximations, to calculate the relative velocity of the particle, v_x, at any point from the place of injection onward, and also to calculate the time during which the particle is being entrained by the gas stream.

We shall examine the inertial range of a particle under the action of an inertia force in the stagnant zone ahead of an obstacle in the region Re ≤ 300.

When a particle is injected into a stationary medium, it is slowed down by the drag force. The value of the relative velocity between the particle and the medium decreases as the particle is decelerated.

The differential equation for deceleration of a particle, allowing for gravity (for ascending and descending streams) is as follows:

$$\frac{\pi d^3}{6} \rho_2 \frac{dv}{dt} = \left(A + \frac{B}{\mathrm{Re}}\right) \frac{\pi d^2}{8} \rho_1 v_0^2 \quad \frac{\pi d^3}{6} \rho_2 g. \quad (12)$$

Multiplying both sides of (12) by dx, and making the appropriate transformations, we obtain

$$dx = \frac{v_0 dv_0}{\alpha v_0^2 + \beta v_0 + g}. \tag{13}$$

Integrating (13) in the range x = 0 (i.e., the time of injection of the particle into the stagnant region) to $x = l_i$, and the relative velocity from v_0 to v_x, we obtain equations which permit an approximate calculation of the inertial range:

a) when $4\alpha g - \beta^2 > 0$

$$l_{i,1} = \int_{v_x}^{v_0} \frac{v_0 dv_0}{\alpha v_0^2 + \beta v_0 + g} = \left| \frac{1}{2\alpha} \ln(\alpha v_0^2 + \beta v_0 + g) - \frac{\beta}{\alpha\sqrt{4\alpha g - \beta^2}} \operatorname{arctg} \frac{2\alpha v_0 + \beta}{\sqrt{4\alpha g - \beta^2}} \right|_{v_x}^{v_0} =$$

$$= \frac{1}{2\alpha} \ln A_1 - B_1 \operatorname{arctg} D_1, \tag{14}$$

where

$$A_1 = \frac{\alpha v_0^2 + \beta v_0 + g}{\alpha v_x^2 + \beta v_x + g}, \quad B_1 = \frac{\beta}{\alpha\sqrt{4\alpha g - \beta^2}},$$

$$C_1 = \sqrt{4\alpha g - \beta^2}, \quad D_1 = \frac{2C_1(\alpha v_0 + \alpha v_x + \beta)}{C_1^2 - (2\alpha v_0 + \beta)(2\alpha v_x + \beta)};$$

b) when $4\alpha g - \beta^2 < 0$

$$l_{i,2} = \int_{v_x}^{v_0} \frac{v_0 dv_0}{\alpha v_0^2 + \beta v_0 + g} = \left| \frac{1}{2\alpha} \ln(\alpha v_0^2 + \beta v_0 + g) - \frac{\beta}{2\alpha\sqrt{\beta^2 - 4\alpha g}} \ln \frac{2\alpha v_0 + \beta - \sqrt{\beta^2 - 4\alpha g}}{2\alpha v_0 + \beta + \sqrt{\beta^2 - 4\alpha g}} \right|_{v_x}^{v_0} =$$

$$= \frac{1}{2\alpha} \ln A_1 - B_2 \ln \frac{(v_0 + q_1)(v_x + q_1)}{(v_0 + q_2)(v_x + q_2)}, \tag{15}$$

where

$$B_2 = \beta[2\alpha^2(q_2 - q_1)]^{-1}, \quad q_{1,2} = -2g(\beta \pm \sqrt{\beta^2 - 4\alpha g})^{-1};$$

c) when $4\alpha g - \beta^2 = 0$

$$l_{i,3} = \int_{v_x}^{v_0} \frac{v_0 dv_0}{\alpha v_0^2 + \beta v_0 + g} =$$

$$= \frac{1}{2\alpha} \ln A_1 - 2\beta \frac{v_0 - v_x}{(2\alpha v_0 + \beta)(2\alpha v_x + \beta)}; \tag{16}$$

d) when $4\alpha g - \beta^2 < 0$ and $(2\alpha v_0 + \beta)^2 < (\beta^2 - 4\alpha g)$

$$l_{i,4} = \int_{v_x}^{v_0} \frac{v_0 dv_0}{\alpha v_0^2 + \beta v_0 + g} = \frac{1}{2\alpha} \ln A_1 +$$

$$+ \frac{\beta}{\alpha C_2} \operatorname{arth} \frac{2\alpha C_2 (v_0 - v_x)}{C_2^2 - (2\alpha v_0 + \beta)(2\alpha v_x + \beta)}, \tag{17}$$

where

$$C_2 = \sqrt{\beta^2 - 4\alpha g};$$

e) when $4\alpha g - \beta^2 < 0$ and $(2\alpha v_0 + \beta)^2 > (\beta^2 - 4\alpha g)$

$$l_{i,5} = \int_{v_x}^{v_0} \frac{v^0 dv_0}{\alpha v_0^2 + \beta v_0 + g} = \frac{1}{2\alpha} \ln A_1 +$$

$$+ \frac{\beta}{\alpha C_2} \operatorname{arcth} \frac{2\alpha C_2 (v_0 - v_x)}{C_2^2 - (2\alpha v_0 + \beta)(2\alpha v_x + \beta)}. \tag{18}$$

For small particles ($d < 10\ \mu$) the force of gravity may be neglected, and then (12) transforms to

$$dx = dv_0/(\alpha v_0 + \beta). \tag{19}$$

Integrating (19) in the same range as (13), we obtain

$$l_{i,6} = \int_{v_x}^{v_0} \frac{dv_0}{\alpha v_0 + \beta} = \gamma_1 \ln \left| \frac{\alpha v_0 + \beta}{\alpha v_x + \beta} \right|, \tag{20}$$

where $\gamma_1 = 1.33\, d\rho_2(A\rho_1)^{-1}$.

In the case in which mdv/dt = 0, the particle acquires a steady sedimentation velocity under the action of gravity.

Particle Sedimentation Velocity and Inertial Range

d, μ	Relaxation time τ, sec	Sedimentation velocity v_s cm/sec		Inertial range l_i, cm	
		according to the Stokes equation	according to (23)	according to the Stokes equation	according to (20)
0.1	$3.1\cdot10^{-8}$	$3\cdot10^{-5}$	—	$32\cdot10^{-5}$	$5\cdot10^{-5}$
0.5	$7.8\cdot10^{-7}$	$76\cdot10^{-5}$	—	$55\cdot10^{-4}$	$30\cdot10^{-4}$
1.0	$3.1\cdot10^{-6}$	$3\cdot10^{-3}$	—	$22\cdot10^{-3}$	$8.17\cdot10^{-3}$
2.0	$1.2\cdot10^{-5}$	$12\cdot10^{-3}$	—	$84\cdot10^{-3}$	$5\cdot10^{-2}$
5.0	$7.8\cdot10^{-5}$	$76\cdot10^{-3}$	—	$55\cdot10^{-2}$	$3.3\cdot10^{-2}$
10.0	$3.1\cdot10^{-4}$	0.3	—	2.17	1.22
20.0	$1.2\cdot10^{-3}$	1.2	1.0	8.4	4.58
50.0	$7.8\cdot10^{-3}$	7.6	3.0	54.6	25.7
100.0	$3.1\cdot10^{-2}$	31.0	20.0	217.0	80.6
200.0	$6.2\cdot10^{-2}$	62.0	39.5	434.0	240.7
500.0	$16\cdot10^{-2}$	160.0	118.0	5460.0	930.0

Using the experimental data of [10, 11], we shall express the value of the coefficient ψ_s at constant velocity in the region $2 < Re \leq 300$:

$$\psi_s = A_s + B_s/Re, \quad (21)$$

where $A_s = 0.60$; $B_s = 33$.

For more accurate calculations of sedimentation velocity in the region $Re \leq 300$ (to accuracy ±0.3–0.8%), we must assume the following values of the constants: a) for the region $100 < Re \leq 300$ $A_s = 0.402$, $B_s = 70$, b) for $10 < Re \leq 100$ 0.89 and 41; c) for $Re \leq 10$ 2.20 and 24.

Substituting into (12) the value of ψ_s from (21), and taking $mdv/dt = 0$, then, we obtain the differential equation of motion of the particle in the region $Re \leq 300$, which we shall transform to the following form:

$$v_0^2 + \alpha_1 v_0 - \beta_1 = 0, \quad (22)$$

where $\alpha_1 = B_s \nu (A_s d)^{-1}$, $\beta_1 = 0.75\ \rho_2 dg (A_s \rho_1)^{-1}$.

From (22) we find the steady sedimentation velocity of the particle

$$v_s = 0.5(-\alpha_1 + \sqrt{\alpha_1^2 + 4\beta_1}). \quad (23)$$

The values of steady sedimentation velocity calculated from (23) agree satisfactorily with the experimental data of [10, 11] in the region $2 < Re \leq 300$.

The table gives values of the steady sedimentation velocity and inertial range of particles, as calculated from (20) and (23). The calculation was performed for particles with density $\rho_2 = 1000$ kg/m^3, $v_0 = 70$ m/sec, pressure $B = 9.81 \cdot 10^4$ N/m^2, viscosity $\eta = 1.8 \times 10^{-5}$ N · sec/m^2. The inertial range of the particle was calculated from (20) with the value $v_x \sim 0$.

As is seen from the table, values of the steady particle sedimentation velocity and inertial range, as calculated from the Stokes equation, are considerably different from those calculated from Eqs. (20) and (23) in the region $Re \leq 300$.

The equations obtained, (14)–(18) and (20), allow determination of the velocity of a particle at any point x from the start of injection. Thus, for example, after appropriate transformations of Eq. (20), we find that the particle velocity v_x at distance $l_{i,n}$ from the point of entry into the stationary medium is

$$v_x = v_0 \exp\left(-\frac{l_{i,n}}{\gamma_1}\right) + A_2\left[\exp\left(-\frac{l_{i,n}}{\gamma_1}\right) - 1\right], \quad (24)$$

where $A_2 = B_s \nu (A_s \alpha_1)^{-1}$.

When $l_{i,n} = 0$, we obtain $v_x = v_0$.

NOTATION

d – particle diameter, m; η – viscosity of medium, N · sec/m^2; $\bar{v}$, $\bar{v}_0$ – absolute and relative particle velocity, m/sec; ρ_1, ρ_2 – density of medium and particle, kg/m^3; t – time, sec; x – coordinate in the direction of motion of the particle and of the stream, m; $\bar{v}_g$ – gas stream velocity, m/sec; v_x – relative particle velocity at distance x from the origin, m/sec; ν – viscosity of medium, m^2/sec; ψ, ψ_s – drag coefficient of medium during decelerating motion and at constant sedimentation velocity; l_e – particle entrainment (acceleration) length, m; l_i – inertial range of particle, m; v_s – steady particle sedimentation velocity, m/sec.

REFERENCES

1. N. A. Fuks, Mechanics of Aerosols [in Russian], Izd. AN SSSR, 1955.

2. L. Landau and E. Lifshits, Mechanics of Continuous Media [in Russian], Gostekhizdat, 1944.

3. I. Serafini, Nat. Adv. Comm. Aeron (NACA), Report no. 1159, 1954.

4. L. S. Klyachko, Otoplenie i ventilyatsiya, no. 5, 1934.

5. A. T. Litvinov, Teploenergetika, no. 5, 1964.

6. S. Goldstein, Proc. Roy. Soc., **123A**, 225, 1929.

7. C. Oseen, Neuere Methoden und Ergebnisse in der Hydrodynamie, Leipzig, 1927.

8. R. Ingebo, Nat. Adv. Comm. Aeron (NACA), Techn. Note, 3762, 1956.

9. A. T. Litvinov, ZhPKh, no. 10, 1965.

10. K. Gunn and G. D. Kinzer, Journ. Meteorol., 6, 243, 1949.

11. C. E. Lapple, Chem. Eng. Handbook, N I, 1018, 1950.

27 November 1965

Institute of Industrial and Sanitary Gas Purification, Moscow